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Oscillatory free convection from an infinite horizontal cylinder

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The boundary layer on a horizontal cylinder caused by free convection, when the temperature of the cylinder is oscillating harmonically, is considered in a way which is analogous to the problem of a cylinder performing high-frequency oscillations along its diameter. It is found that, outside the thin boundary layer on the cylinder, a steady flow is induced. An outer steady boundary layer, matched with the inner boundary layer, is required in order to be able to satisfy the conditions which apply at large distances from the cylinder.

1. Introduction

In this paper we consider the problem of the boundary layer on a fixed cylinder, on which the temperature is oscillating harmonically with frequency w about a mean temperature T_0, the temperature of the surrounding fluid. The motion is caused by the action of oscillating buoyant body forces on the fluid near the cylinder. This gives a situation analogous to that of a cylinder performing harmonic oscillations along an axis perpendicular to its generators. So the method of solution that we adopt is to follow the work of Schlichting (1932) and Riley (1965) on the boundary layer on an oscillating cylinder. Schlichting develops a solution in powers of U_∞/wd, which is assumed small; here U_∞ and d are a typical velocity and length respectively. The assumption implicit in this method is that the non-linear terms in the boundary-layer equations are of smaller order than the linear terms, and can be neglected for a first approximation. The justification for doing this is discussed by Stuart (1963, pp. 349–56). Schlichting found that he could not make the tangential component of velocity $u \to 0$ at the edge of the boundary layer, and a steady velocity of $O(U_\infty^2/wd)$ persisted outside this boundary layer, of thickness $O(\nu/w)^{\frac{1}{2}}$; ν is the kinematic viscosity. Stuart (1966) explains that the Reynolds number $R_s = U_\infty^2/w\nu$ of the steady outer flow is the parameter which is important in determining how $u \to 0$ outside the inner boundary layer. For $R_s \gg 1$ the outer flow is again governed by the boundary-layer equations, and it is at the outer edge of this outer boundary layer that u finally becomes zero. Riley matches a solution which is valid in the outer boundary layer with the solution which holds in the inner boundary layer. It is this matching procedure which determines an unknown constant appearing in the inner layer solution.

In the problem under consideration here, the temperature on the cylinder T_w is given by $T_w - T_0 = aT_0 \cos wt$, where T_0 is the temperature at large distances from the cylinder. As in all problems of free convection there is no obvious typical velocity scale, and we find that the appropriate velocity scale in this case is $g\beta aT_0/w$. So the frequency parameter ϵ which corresponds to the small parameter in Schlichting's work is $g\beta aT_0/w^2R$, where g is the acceleration due to gravity, β is the coefficient of the thermal expansion, and R is a typical radius of the cylinder. As in the case of the oscillating cylinder, we find that outside an inner boundary layer, of thickness $O(K/w)^{\frac{1}{2}}$, we need an outer boundary layer, of thickness $O(\epsilon^{-1}(K/w)^{\frac{1}{2}})$, to be matched with the inner boundary layer, in order to be able to satisfy all the required boundary conditions; K is the thermometric conductivity. We assume that the system has been oscillating for a long time, so that transients have died out, and we can look for solutions which are varying only harmonically with time. A solution in the inner boundary layer is developed in powers of $\epsilon^{\frac{1}{2}}$, which is assumed to be small. Although only integral powers of ϵ are suggested by the form of the equations in the inner layer, we find that it is necessary to include terms of $O(\epsilon^{\frac{1}{2}})$ in the inner layer expansion in order to be able to solve the equations which arise in the outer boundary layer. We find that we cannot make $u \to 0$, and $T \to T_0$ at the outer edge of the inner boundary layer, and a steady tangential component of velocity of $O(\epsilon g\beta aT_0/w)$ and a steady temperature difference of $O(\epsilon aT_0)$ persist outside this boundary layer. As pointed out by Riley, once we admit the presence of an outer boundary layer, there is no reason to suppose that $(T - T_0)$ and u should remain finite at the outer edge of the inner boundary layer. This introduces an arbitrariness into the inner layer solution, as we apply boundary conditions only at the cylinder. The conditions at the outer edge of the inner layer are determined by matching the inner layer solution with the solution which is valid in the outer layer. The appropriate Reynolds number R_s of the steady outer flow is $(g\beta aT_0)^2/w^3\nu$, and with $R_s \gg 1$ the equations governing the outer flow are again the boundary-layer equations. In the outer boundary layer the nonlinear convective terms are of the same order as the diffusive and buoyancy terms, and so the inner layer expansion breaks down. Following the method of Riley, a solution of the equations governing the steady outer flow is obtained in the form of series expansions, analogous to the Blasius series, about the stagnation points of the steady outer flow. This solution in the outer layer is matched with the solution in the inner layer.

2. Equations of motion

The fluid is assumed to be almost incompressible, so that changes in density are important only in producing buoyancy forces. The kinematic viscosity ν, and the thermometric conductivity K are taken as constants, and in the energy equation work done by the pressure, and the effects of viscous dissipation are neglected. It is found that the velocity is $O(g\beta aT_0/w)$ in the inner boundary layer, and, with the temperature on the cylinder T_w given by $T_w - T_0 = aT_0 \cos wt$, these assumptions are justifiable when

$$(g\beta aT_0/wc)^2 \ll a \ll 1 \quad \text{and} \quad wR/c \ll 1$$

(Lighthill 1963, pp. 9–13; Whitham 1963, pp. 126–7); here c is the velocity of sound in the fluid.

The infinite cylinder is fixed with its axis horizontal, and so the problem is essentially two-dimensional. The co-ordinate x is defined as the distance measured along the surface of the cylinder, the lowest point being the origin $x = 0$; and the co-ordinate y is defined to be the distance measured normally outwards from the cylinder. The angle α is taken to be the angle made by the outward normal with the downward vertical.

The boundary-layer equations, with the co-ordinate system specified above, are (Goldstein 1938, pp. 610–13)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta(T-T_0)\sin\alpha+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

with boundary conditions

$$u = v = 0,$$
$$T-T_0 = aT_0\cos wt \quad \text{on} \quad y = 0,$$
$$u\to 0, \quad T\to T_0 \quad \text{as} \quad y\to\infty.$$

u and v are the velocity components associated with the x and y directions respectively, and T is the temperature of the fluid. The appropriate Reynolds number of the inner boundary layer is $(g\beta aT_0)R/w\nu$ and of the outer boundary layer is $(g\beta aT_0)^2/w^3\nu$. In these boundary-layer equations, the largest terms neglected are $O(\text{Reynolds number})^{\frac{1}{2}}$. We must also impose certain conditions on the curvature k of the cylinder to make the boundary-layer simplifications; these are

$$k\delta \ll 1, \quad R\delta(dk/dx) \ll 1,$$

where δ is a measure of the boundary-layer thickness, and is $O(K/w)^{\frac{1}{2}}$ for the inner boundary layer, and $O(\epsilon^{-1}(K/w)^{\frac{1}{2}})$ for the outer boundary layer.

3. Solution in the inner layer

To develop a solution in the inner layer we follow the method of Schlichting, which is discussed by Stuart (1963, pp. 349–56 and pp. 382–8). The nonlinear convective terms can be assumed to be of smaller order than the linear terms in the boundary-layer equations (2) and (3), when the frequency parameter $\epsilon = g\beta aT_0/w^2R$ is small.

From the continuity equation (1) we can define a streamfunction ψ by

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}.$$

We introduce non-dimensional variables ξ, τ, η, θ and f as

$$\xi = x/R, \quad \tau = wt, \quad \eta = (w/2K)^{\frac{1}{2}}y, \tag{4}$$

$$\frac{T-T_0}{aT_0} = \theta(\xi,\eta,\tau), \tag{5}$$

$$\psi = \left(\frac{2K}{w}\right)^{\frac{1}{2}}\left(\frac{g\beta aT_0}{w}\right)f(\xi,\eta,\tau) \tag{6}$$

and $\sigma = \nu/K$ is the Prandtl number. Writing $\sin\alpha \equiv S(\xi)$, equations (2) and (3) then become

$$\frac{\partial^2 f}{\partial\eta\,\partial\tau} - \frac{\sigma}{2}\frac{\partial^3 f}{\partial\eta^3} - \theta S(\xi) = \epsilon\left(\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\xi}\right), \tag{7}$$

$$\frac{\partial\theta}{\partial\tau} - \frac{1}{2}\frac{\partial^2\theta}{\partial\eta^2} = \epsilon\left(\frac{\partial f}{\partial\xi}\frac{\partial\theta}{\partial\eta} - \frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi}\right), \tag{8}$$

with boundary conditions on $\eta = 0$ that

$$\theta = \cos\tau \quad \text{and} \quad \partial f/\partial\eta = \partial f/\partial\xi = 0. \tag{9}$$

We would like to apply the boundary conditions that $\theta \to 0$ and $\partial f/\partial\eta \to 0$ as $\eta \to \infty$, but, as in the case of the oscillating cylinder, we find that we cannot satisfy these conditions as well as the conditions that we must apply on $\eta = 0$. For this reason we regard this solution as valid only in an inner boundary layer, beyond which we introduce an outer boundary layer. The resulting arbitrariness of the inner solution is resolved by matching the solution at the outer edge of the inner layer with the solution at the inner edge of the outer layer. We now proceed with the inner layer solution on the assumption that f and θ must be only algebraically large as $\eta \to \infty$.

Equations (7) and (8) suggest expanding f and θ in powers of ϵ, but we find that we cannot solve the equations which arise in the outer layer when we do. An expansion in powers of $\epsilon^{\frac{1}{2}}$ is found necessary. We expand $f(\xi,\eta,\tau)$ and $\theta(\xi,\eta,\tau)$ in the form

$$f = S(\xi) f_0(\eta)\, e^{i\tau} + S\frac{dS}{d\xi}\left[\epsilon^{\frac{1}{2}} f_1(\xi,\eta) + \epsilon(f_{2s}(\eta) + f_{22}(\eta)\, e^{2i\tau})\right] + O(\epsilon^{\frac{3}{2}}), \tag{10}$$

$$\theta = \theta_0(\eta)\, e^{i\tau} + \frac{dS}{d\xi}\left[\epsilon^{\frac{1}{2}}\theta_1(\xi,\eta) + \epsilon(\theta_{2s}(\eta) + \theta_{22}(\eta)\, e^{2i\tau})\right] + O(\epsilon^{\frac{3}{2}}), \tag{11}$$

where only the real part is to be taken. The dependence of the terms of $O(\epsilon)$ on ξ follows naturally from θ_0 and f_0, but the equations for f_1 and θ_1 are not connected with those for f_0 and θ_0, so we have to allow for an arbitrary dependence on ξ in θ_1 and f_1. We put the expansions (10) and (11) into equations (7) and (8), and equate powers of ϵ. The solutions of the resulting equations which satisfy the boundary conditions on $\eta = 0$ that

$$\theta_0 = 1, \quad \theta_1 = \theta_{2s} = \theta_{22} = 0,$$

$$f_0 = f_0' = f_1 = f_1' = f_{2s} = f_{2s}' = f_{22} = f_{22}' = 0,$$

are, provided $\sigma \neq 1$,

$$\theta_0 = \exp\{-(1+i)\eta\}, \tag{12}$$

$$f_0 = \frac{(1+i)}{2(\sigma-1)}\left[(1-\sigma^{\frac{1}{2}}) + \sigma^{\frac{1}{2}}\exp\{-(1+i)\eta\sigma^{-\frac{1}{2}}\} - \exp\{-(1+i)\eta\}\right], \tag{13}$$

$$\theta_1 = A(\xi)\,\eta, \tag{14}$$

$$f_1 = \frac{1}{\sigma}\left[B(\xi)\,\eta^2 - \frac{A(\xi)}{12}\eta^4\right] \tag{15}$$

((14) and (15) are the most general solutions of the equations of $O(\epsilon^{\frac{1}{2}})$ satisfying the boundary conditions),

$$
\begin{aligned}
\theta_{2s} = \frac{\sigma^2}{(\sigma+1)^2(\sigma-1)} & \exp\{-\eta(1+\sigma^{-\frac{1}{2}})\}\cos((1-\sigma^{-\frac{1}{2}})\eta) \\
& -\frac{\sigma^{\frac{3}{2}}}{2(\sigma+1)^2}\exp\{-\eta(1+\sigma^{-\frac{1}{2}})\}\sin(\eta(1-\sigma^{-\frac{1}{2}})) \\
& -\frac{1}{4(\sigma-1)}\exp\{-2\eta\}+\frac{1}{2(\sigma^{\frac{1}{2}}+1)}\exp\{-\eta\}\sin\eta \\
& -\frac{1+3\sigma}{4(\sigma+1)^2}+C\eta,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\theta_{22} = \frac{1}{2(\sigma^{\frac{1}{2}}+1)} & \exp\{-(1+i)\eta\} \\
& +\frac{\sigma^{\frac{3}{2}}}{2(\sigma-1)(1+2\sigma^{\frac{1}{2}}-\sigma)}\exp\{-(1+i)(1+\sigma^{-\frac{1}{2}})\eta\} \\
& -\frac{1}{4(\sigma-1)}\exp\{-2(1+i)\eta\} \\
& -\frac{3+7\sigma^{\frac{1}{2}}}{4(1+2\sigma^{\frac{1}{2}}-\sigma)(\sigma^{\frac{1}{2}}+1)}\exp\{-(1+i)\eta\sqrt{2}\},
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
f_{2s} = & \frac{1}{4(\sigma-1)(\sigma^{\frac{1}{2}}+1)}\exp\{-\eta\sigma^{-\frac{1}{2}}\}(\sin(\eta\sigma^{-\frac{1}{2}})-\cos(\eta\sigma^{-\frac{1}{2}})) \\
& -\frac{1}{4(\sigma-1)(\sigma^{\frac{1}{2}}+1)}\exp\{-\eta\}(\sin\eta-\cos\eta) \\
& -\frac{\sigma^{\frac{1}{2}}}{8(\sigma-1)^2}\exp\{-2\eta\sigma^{-\frac{1}{2}}\}-\frac{\sigma+1}{16(\sigma-1)^2\sigma}\exp\{-2\eta\} \\
& -E\exp\{-\eta(1+\sigma^{-\frac{1}{2}})\}\sin(\eta(1-\sigma^{-\frac{1}{2}})) \\
& -F\exp\{-\eta(1+\sigma^{-\frac{1}{2}})\}\cos(\eta(1-\sigma^{-\frac{1}{2}})) \\
& -\frac{C\eta^4}{12\sigma}+\frac{1+3\sigma}{12\sigma(\sigma+1)^2}\eta^3+\frac{D\eta^2}{2}+G\eta+H,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
f_{22} = A_2+B_2 & \exp\{-(1+i)\eta\sqrt{2}\,\sigma^{-\frac{1}{2}}\}+L\exp\{-(1+i)\eta\} \\
& +M\exp\{-(1+i)\eta\sigma^{-\frac{1}{2}}\}+N\exp\{-(1+i)\eta(1+\sigma^{-\frac{1}{2}})\} \\
& +P\exp\{-2(1+i)\eta\}+Q\exp\{-(1+i)\eta\sqrt{2}\},
\end{aligned} \tag{19}
$$

where C and D are constants which are undetermined at this stage, and

$$
E = \frac{(\sigma^3+2\sigma+1)(4\sigma^{\frac{3}{2}}+14\sigma+4\sigma^{\frac{1}{2}}-\sigma^2-1)}{4(\sigma^{\frac{1}{2}}+1)(\sigma+1)^5(\sigma-1)},
$$

$$
F = \frac{(\sigma^3+2\sigma+1)(4\sigma^{\frac{3}{2}}-14\sigma+4\sigma^{\frac{1}{2}}+\sigma^2+1)}{4(\sigma^{\frac{1}{2}}-1)(\sigma-1)(\sigma+1)^5},
$$

$$
G = \frac{1}{2\sigma^{\frac{1}{2}}(\sigma^{\frac{1}{2}}+1)^2}-\frac{(3\sigma+1)}{8\sigma(\sigma-1)^2}-\frac{(\sigma^2-6\sigma+1)(\sigma^3+2\sigma+1)}{2\sigma^{\frac{1}{2}}(\sigma+1)^4(\sigma-1)^2},
$$

$$H = \frac{(\sigma^3+2\sigma+1)(4\sigma^{\frac{3}{2}}-14\sigma+4\sigma^{\frac{1}{2}}+\sigma^2+1)}{4(\sigma+1)^5(\sigma-1)(\sigma^{\frac{1}{2}}-1)} + \frac{(2\sigma-\sigma^{\frac{1}{2}}+1)}{16\sigma(\sigma-1)(\sigma^{\frac{1}{2}}-1)},$$

$$L = -\left(\frac{1+i}{4}\right)\frac{1}{(\sigma-1)(\sigma^{\frac{1}{2}}+1)},$$

$$M = \left(\frac{1+i}{4}\right)\frac{1}{(\sigma-1)(\sigma^{\frac{1}{2}}+1)},$$

$$N = \frac{(\sigma^3-2\sigma^2+4\sigma^{\frac{3}{2}}-4\sigma+1)}{(1+\sigma^{\frac{1}{2}})(\sigma^2-6\sigma+1)(\sigma-1)^2}\left(\frac{1+i}{4}\right),$$

$$P = \left(\frac{1+i}{32}\right)\frac{1}{(\sigma-1)(2\sigma-1)},$$

$$Q = \left(\frac{1+i}{32}\right)\frac{(3+7\sigma^{\frac{1}{2}})\sqrt{2}}{(1+2\sigma^{\frac{1}{2}}-\sigma)(\sigma^{\frac{1}{2}}+1)(\sigma-1)},$$

$$A_2 = \left(\frac{1+i}{32}\right)\frac{[(3-8\sqrt{2})\sigma^{\frac{3}{2}}-(13+9\sqrt{2})\sigma-(1+14\sqrt{2})\sigma^{\frac{1}{2}}-9+3\sqrt{2}]}{(\sigma^{\frac{1}{2}}+1)^3(\sqrt{2}\sigma^{\frac{1}{2}}+1)(\sigma^{\frac{1}{2}}+1+\sqrt{2})(\sigma^{\frac{1}{2}}-1+\sqrt{2})},$$

$$B_2 = \left(\frac{1+i}{4\sqrt{2}}\right)\frac{1}{(\sigma^{\frac{1}{2}}+1)^2} - \left(\frac{1+i}{32}\right)\frac{\sqrt{2}}{(\sigma-1)}\frac{(7\sigma-2+13\sigma^{\frac{3}{2}}-4\sigma^{\frac{1}{2}})}{(2\sigma-1)(\sigma^{\frac{1}{2}}+1)(1+2\sigma^{\frac{1}{2}}-\sigma)} - \left(\frac{1+i}{4}\right)\frac{(\sigma^3-2\sigma^2+4\sigma^{\frac{3}{2}}-4\sigma+1)}{\sqrt{2}(\sigma-1)^2(\sigma^2-6\sigma+1)}.$$

For the case when $\sigma = 1$, the solution is

$$\theta_0 = \exp\{-(1+i)\eta\}, \tag{20}$$

$$f_0 = \frac{i}{2}\eta\exp\{-(1+i)\eta\} + \left(\frac{1+i}{4}\right)(\exp\{-(1+i)\eta\}-1), \tag{21}$$

$$\theta_1 = A(\xi)\eta, \tag{22}$$

$$f_1 = B(\xi)\eta^2 - A(\xi)\frac{\eta^4}{12}, \tag{23}$$

$$\theta_{2s} = \tfrac{1}{8}(\eta e^{-2\eta}+2e^{-2\eta}+8C\eta-2+2e^{-\eta}\sin\eta), \tag{24}$$

$$\theta_{22} = \tfrac{1}{4}\exp\{-(1+i)\eta\} - \tfrac{5}{8}\exp\{-(1+i)\eta\sqrt{2}\} + \left(\frac{1+i}{8}\right)\eta\exp\{-2(1+i)\eta\} + \tfrac{3}{8}\exp\{-2(1+i)\eta\}, \tag{25}$$

$$f_{2s} = \frac{\eta^3}{12} - \frac{C\eta^4}{12} + \frac{D\eta^2}{2} + \frac{11}{64} - \frac{\eta}{8}e^{-\eta}\cos\eta - \tfrac{11}{64}e^{-2\eta} - \frac{\eta^2}{16}e^{-2\eta} - \tfrac{7}{32}\eta e^{-2\eta}, \tag{26}$$

$$f_{22} = \left(\frac{1+i}{128}\right)(13-11\sqrt{2}) + \frac{i}{8}\eta\exp\{-(1+i)\eta\} + \tfrac{11}{128}(1+i)\sqrt{2}\exp\{-(1+i)\sqrt{2}\eta\} - \frac{5i}{32}\eta\exp\{-(1+i)\eta\sqrt{2}\} - \frac{i}{32}\eta\exp\{-2(1+i)\eta\} - \tfrac{13}{128}(1+i)\exp\{-2(1+i)\eta\}. \tag{27}$$

At the outer edge of the inner boundary layer (as $\eta \to \infty$)

$$T - T_0 \sim aT_0 \frac{dS}{d\xi}\left[\epsilon^{\frac{1}{2}} A(\xi)\,\eta + \epsilon\left(C\eta - \frac{1+3\sigma}{4(\sigma+1)^2}\right)\right] + O(\epsilon^{\frac{3}{2}}), \tag{28}$$

$$\psi + \frac{g\beta aT_0}{w}\left(\frac{2K}{w}\right)^{\frac{1}{2}} \frac{S(\xi)}{\sqrt{2}(\sigma^{\frac{1}{2}}+1)} \cos(\tau + \tfrac{1}{4}\pi)$$
$$\sim \left(\frac{g\beta aT_0}{w}\right)\left(\frac{2K}{w}\right)^{\frac{1}{2}} S \frac{dS}{d\xi}\left[\frac{\epsilon^{\frac{1}{2}}}{\sigma}\left(B(\xi)\,\eta^2 - A(\xi)\frac{\eta^4}{12}\right)\right.$$
$$\left. + \epsilon\left(-\frac{C\eta^4}{12\sigma} + \frac{1+3\sigma}{12\sigma(\sigma+1)^2}\eta^3 + D\frac{\eta^2}{2} + G\eta + H + A_2 e^{2i\tau}\right)\right] + O(\epsilon^{\frac{3}{2}}) \tag{29}$$

and

$$u \sim \left(\frac{g\beta aT_0}{w}\right) S \frac{dS}{d\xi}\left[\frac{\epsilon^{\frac{1}{2}}}{\sigma}\left(2B(\xi)\,\eta - A(\xi)\frac{\eta^3}{3}\right)\right.$$
$$\left. + \epsilon\left(-\frac{C\eta^3}{3\sigma} + \frac{1+3\sigma}{4\sigma(1+\sigma)^2}\eta^2 + D\eta + G\right)\right] + O(\epsilon^{\frac{3}{2}}). \tag{30}$$

The asymptotic forms for ψ and $(T - T_0)$ hold also when $\sigma = 1$, in which case

$$A_2 = \left(\frac{1+i}{128}\right)(13 - 11\sqrt{2}).$$

From (28) and (30) we see that we cannot satisfy the boundary condition that $u \to 0$ and $T \to T_0$ as $\eta \to \infty$ even by setting $A(\xi) = B(\xi) = 0$ and $C = D = 0$. Thus the expansions (10) and (11) are valid only in a region of thickness $O(K/w)^{\frac{1}{2}}$, and the asymptotic forms for the streamfunction ψ and the temperature difference $(T - T_0)$ must be regarded as the inner conditions for the outer boundary layer.

4. Solution in the outer layer

The steady flow outside the inner boundary layer depends on the Reynolds number of this outer flow (Stuart 1963, pp. 384–5). Here the appropriate Reynolds number is $R_s = (g\beta aT_0)^2/w^3\nu$, and with $R_s \gg 1$ the equations governing this outer flow are again the boundary-layer equations (1), (2) and (3). The thickness of this outer layer is $O(\epsilon^{-1}(K/w)^{\frac{1}{2}})$ and it is at the outer edge of this boundary layer that $u \to 0$ and $T \to T_0$.

In the outer layer the nonlinear convective terms are of the same order as the diffusive terms occurring in (2) and (3). Thus the method of expanding ψ and T in powers of $\epsilon^{\frac{1}{2}}$ in the inner layer solution breaks down because the assumption implicit in this method is that the nonlinear terms are of smaller order than the linear terms in equations (2) and (3).

We develop the outer layer solution on the assumption that $A(\xi) \equiv 0$ and $C \equiv 0$; a justification for this is given in appendices II and III.

In the outer layer the convective terms are of the same order as the diffusive and buoyancy terms. This suggests the definition of non-dimensional variables as

$$\frac{T - T_0}{aT_0} = \epsilon\bar{\theta}(\xi, \bar{\eta}, \tau), \tag{31}$$

$$\psi + \left(\frac{g\beta aT_0}{w}\right)\left(\frac{K}{w}\right)^{\frac{1}{2}} \frac{S(\xi)}{(1+\sigma^{\frac{1}{2}})} \cos(\tau + \tfrac{1}{4}\pi) = \epsilon^{-\frac{1}{2}}\left(\frac{K}{w}\right)^{\frac{1}{2}}\left(\frac{g\beta aT_0}{w}\right)\bar{f}(\xi, \bar{\eta}, \tau), \tag{32}$$

with

$$\bar{\eta} = \epsilon^{\frac{1}{2}}(w/K)^{\frac{1}{2}} y, \tag{33}$$

and with ξ and τ given by (4). Equations (2) and (3) then become

$$\frac{\partial^2\bar{f}}{\partial\bar{\eta}\,\partial\tau}+\epsilon\left(\frac{\partial\bar{f}}{\partial\bar{\eta}}\frac{\partial^2\bar{f}}{\partial\bar{\eta}\,\partial\xi}-\frac{\partial\bar{f}}{\partial\xi}\frac{\partial^2\bar{f}}{\partial\bar{\eta}^2}-S(\xi)\,\bar{\theta}-\sigma\frac{\partial^3\bar{f}}{\partial\bar{\eta}^3}\right)+\epsilon^{\frac{3}{2}}\frac{dS}{d\xi}\frac{\cos\left(\tau+\frac{1}{4}\pi\right)}{(1+\sigma^{\frac{1}{2}})}\frac{\partial^2\bar{f}}{\partial\bar{\eta}^2}=0, \tag{34}$$

$$\frac{\partial\bar{\theta}}{\partial\tau}+\epsilon\left(\frac{\partial\bar{f}}{\partial\bar{\eta}}\frac{\partial\bar{\theta}}{\partial\xi}-\frac{\partial\bar{f}}{\partial\xi}\frac{\partial\bar{\theta}}{\partial\bar{\eta}}-\frac{\partial^2\bar{\theta}}{\partial\bar{\eta}^2}\right)+\epsilon^{\frac{3}{2}}\frac{\cos\left(\tau+\frac{1}{4}\pi\right)}{(\sigma^{\frac{1}{2}}+1)}\frac{dS}{d\xi}\frac{\partial\bar{\theta}}{\partial\bar{\eta}}=0. \tag{35}$$

The boundary conditions on $\bar{f}$ and $\bar{\theta}$ are that $\partial\bar{f}/\partial\bar{\eta}\to 0$, $\bar{\theta}\to 0$ as $\bar{\eta}\to\infty$, and the inner condition, given by the requirement that the solution in the inner layer as $\eta\to\infty$ should match with the solution in the outer layer as $\bar{\eta}\to 0$, is

$$\bar{f}\sim S\frac{dS}{d\xi}\left[\frac{B(\xi)}{\sigma\sqrt{2}}\bar{\eta}^2+\frac{(1+3\sigma)}{24\sigma(1+\sigma)^2}\bar{\eta}^3\right]+\epsilon^{\frac{1}{2}}S\frac{dS}{d\xi}D\frac{\bar{\eta}^2}{\sqrt{2}}+O(\epsilon), \tag{36}$$

$$\bar{\theta}\sim-\frac{dS}{d\xi}\left(\frac{1+3\sigma}{4(\sigma+1)^2}\right)+O(\epsilon), \tag{37}$$

near $\bar{\eta}=0$.

We can now expand $\bar{f}$ and $\bar{\theta}$ in series in $\epsilon^{\frac{1}{2}}$

$$\bar{f}=\bar{f}_0+\epsilon^{\frac{1}{2}}\bar{f}_1+\epsilon\bar{f}_2+\dots, \tag{38}$$

$$\bar{\theta}=\bar{\theta}_0+\epsilon^{\frac{1}{2}}\bar{\theta}_1+\epsilon\bar{\theta}_2+\dots. \tag{39}$$

We put these expansions in equations (34) and (35) and equate powers of ϵ. The terms of $O(1)$ give the equations

$$\frac{\partial\bar{\theta}_0}{\partial\tau}=0,\quad \frac{\partial^2\bar{f}_0}{\partial\tau\,\partial\bar{\eta}}=0,$$

and using (36) we have that

$$\bar{\theta}_0=\bar{\theta}_0(\xi,\bar{\eta}),\quad \bar{f}_0=\bar{f}_0(\xi,\bar{\eta}). \tag{40}$$

Equating terms of $O(\epsilon^{\frac{1}{2}})$ gives

$$\frac{\partial\bar{\theta}_1}{\partial\tau}=0,\quad \frac{\partial^2\bar{f}_1}{\partial\bar{\eta}\,\partial\tau}=0$$

and, as before, using (36) we have that

$$\bar{\theta}_1=\bar{\theta}_1(\xi,\bar{\eta}),\quad \bar{f}_1=\bar{f}_1(\xi,\bar{\eta}).$$

Equating terms of $O(\epsilon)$ in (34) and (35) gives the equations

$$\frac{\partial^2\bar{f}_2}{\partial\bar{\eta}\,\partial\tau}=\sigma\frac{\partial^3\bar{f}_0}{\partial\bar{\eta}^3}+S(\xi)\,\bar{\theta}_0+\frac{\partial\bar{f}_0}{\partial\xi}\frac{\partial^2\bar{f}_0}{\partial\bar{\eta}^2}-\frac{\partial\bar{f}_0}{\partial\bar{\eta}}\frac{\partial^2\bar{f}_0}{\partial\bar{\eta}\,\partial\xi}, \tag{41}$$

$$\frac{\partial\bar{\theta}_2}{\partial\tau}=\frac{\partial^2\bar{\theta}_0}{\partial\bar{\eta}^2}+\frac{\partial\bar{\theta}_0}{\partial\bar{\eta}}\frac{\partial\bar{f}_0}{\partial\xi}-\frac{\partial\bar{\theta}_0}{\partial\xi}\frac{\partial\bar{\theta}_0}{\partial\bar{\eta}}. \tag{42}$$

From (40) the right-hand sides of equations (41) and (42) are independent of τ, and we can integrate them once with respect to τ. Since we are looking only for solutions which vary harmonically with time we must have

$$\sigma\frac{\partial^3\bar{f}_0}{\partial\bar{\eta}^3}+S(\xi)\,\bar{\theta}_0+\frac{\partial\bar{f}_0}{\partial\xi}\frac{\partial^2\bar{f}_0}{\partial\bar{\eta}^2}-\frac{\partial\bar{f}_0}{\partial\bar{\eta}}\frac{\partial^2\bar{f}_0}{\partial\xi\,\partial\bar{\eta}}=0, \tag{43}$$

$$\frac{\partial^2\bar{\theta}_0}{\partial\bar{\eta}^2}+\frac{\partial\bar{\theta}_0}{\partial\bar{\eta}}\frac{\partial\bar{f}_0}{\partial\xi}-\frac{\partial\bar{\theta}_0}{\partial\xi}\frac{\partial\bar{f}_0}{\partial\bar{\eta}}=0, \tag{44}$$

which, to first order, are the equations governing the flow in the outer region. They become in dimensional variables

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta(T-T_0)\,S(\xi)+\nu\frac{\partial^2 u}{\partial y^2},$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial y^2}.$$

The boundary conditions on $\bar{\theta}_0$ and $\bar{f}_0$ are

$$\bar{\theta}_0 \to 0,\ \partial\bar{f}_0/\partial\bar{\eta}\to 0 \quad \text{as} \quad \bar{\eta}\to\infty, \tag{45}$$

and from the matching conditions (36) and (37)

$$\bar{f}_0 \sim S\frac{dS}{d\xi}\left[\frac{B(\xi)\,\bar{\eta}^2}{\sigma\sqrt{2}}+\frac{(1+3\sigma)}{24\sigma(1+\sigma)^2}\bar{\eta}^3\right], \tag{46}$$

$$\bar{\theta}_0 \sim -\frac{dS}{d\xi}\left(\frac{1+3\sigma}{4(\sigma+1)^2}\right) \quad \text{near} \quad \bar{\eta} = 0. \tag{47}$$

The method of solution of (43) and (44) is to expand $\bar{f}_0$ and $\bar{\theta}_0$ in series, analogous to the Blasius series, about the stagnation points of the steady flow in the outer boundary layer. If ξ_0 is the point where $dS/d\xi = 0$, we put $\xi_1 = \xi-\xi_0$ and expand $\bar{f}_0$ and $\bar{\theta}_0$ in series about the point $\xi_1 = 0$. If we assume that the cylinder is symmetric about $\xi = \xi_0$, then we can take the expansion of $S(\xi_1)$ in powers of ξ_1 in the form

$$S(\xi_1) = 1-a_1(\tfrac{1}{2}\xi_1^2)+a_2(\tfrac{1}{4}\xi_1^4)+O(\xi_1^6),$$

then

$$\frac{dS}{d\xi_1} = -a_1\xi_1+a_2\xi_1^3+O(\xi_1^5)$$

and

$$S\frac{dS}{d\xi_1} = -a_1\xi_1+(a_2+\tfrac{1}{2}a_1^2)\,\xi_1^3+O(\xi_1^5).$$

This suggests expanding $\bar{\theta}_0$ and $\bar{f}_0$ as

$$\bar{\theta}_0 = a_1\xi_1 g_0(s)-(a_2 g_1(s)+a_1^2 g_{10}(s))\,\xi_1^3+O(\xi_1^5), \tag{48}$$

$$\bar{f}_0 = \lambda[h_0(s)\,\xi_1-([a_2/a_1]\,h_1(s)+a_1 h_{10}(s))\,\xi_1^3+O(\xi_1^5)], \tag{49}$$

where $\lambda^4 = a_1$ and $s = \lambda\bar{\eta}$.

Near $\xi_1 = 0$, the steady tangential component of velocity in the outer layer must be directed away from the origin, $\xi_1 = 0$, for a Blasius series type solution to be applicable. This suggests an expansion of $B(\xi_1)$ in the form

$$B(\xi_1) = B_0+B_1\xi_1^2+O(\xi_1^4). \tag{50}$$

Putting the expansions (48) and (49) into equations (43) and (44), and equating powers of ξ_1 gives the system of equations:

$$\sigma h_0'''+g_0+h_0h_0''-h_0'^2 = 0, \tag{51}$$

$$g_0''+h_0g_0'-h_0'g_0 = 0, \tag{52}$$

$$\sigma h_1'''+g_1+h_0h_1''-4h_0'h_1'+3h_0''h_1 = 0, \tag{53}$$

$$g_1''+3g_0'h_1+h_0g_1'-g_0h_1'-3h_0'g_1 = 0, \tag{54}$$

$$\sigma h_{10}'''+\tfrac{1}{2}g_0+g_{10}+h_0h_{10}''-4h_0'h_{10}'+3h_{10}h_0'' = 0, \tag{55}$$

$$g_{10}''+3g_0'h_{10}+h_0g_{10}'-g_0h_{10}'-3h_0'g_{10} = 0, \tag{56}$$

with boundary conditions, from (45), (46) and (47), as

$$h_0(0) = h_0'(0) = h_0(\infty) = 0,$$

$$g_0(0) = \frac{1+3\sigma}{4(\sigma+1)^2}, \quad g_0(\infty) = 0,$$

$$h_1(0) = h_1'(0) = h_1'(\infty) = 0,$$

$$g_1(0) = \frac{1+3\sigma}{4(\sigma+1)^2}, \quad g_1(\infty) = 0,$$

$$h_{10}(0) = h_{10}'(0) = h_{10}'(\infty) = 0,$$

$$g_{10}(0) = 0, \quad g_{10}(\infty) = 0.$$

Dashes denote differentiation with respect to s.

Equations (51) and (52) explain why we had to include terms of $O(\epsilon^{\frac{1}{2}})$ in the inner layer expansion. Without the terms in $\epsilon^{\frac{1}{2}}$, the matching condition on $\bar{f}_0$ would be

$$\bar{f}_0 \sim S\frac{dS}{d\xi}\frac{1+3\sigma}{24\sigma(1+\sigma)^2}\frac{s^3}{\lambda^3}$$

near $s = 0$, which, in turn, would mean that $h_0''(0) = 0$. With this extra condition on $h_0(s)$ we cannot obtain a solution to equations (51) and (52) such that $h_0'(s)$ and $g_0(s) \to 0$ as $s \to \infty$.

Equations (51) to (56) have to be solved numerically, and solutions to equations (51) and (52) for various values of σ, and to equations (53) to (56) for the case $\sigma = 1$, are given in appendix I. The difficulty in the numerical integration of the nonlinear equations (51) and (52) was to find the values of $h_0''(0)$ and $g_0'(0)$ before a marching solution could be performed from $s = 0$. The values of $h_0''(0)$ and $g_0'(0)$ were found by using the 'Haselgrove 2-Point Boundary Value Program', which is stored on magnetic tape in Manchester University's Atlas Computer. This program is supplied with guesses for the unknown boundary values at both $s = 0$ and $s = \infty$ (which is chosen to be at a suitably large finite value of s), and, by integrating forwards from $s = 0$, and backwards from $s = \infty$, tries to 'fit' at an intermediate value. The 'fitting' is done by adjusting the unknown boundary values so that the differences in the functions obtained by the forward and the backward integrations are as small as possible.

The first two constants B_0 and B_1 in the expansion of $B(\xi_1)$ near $\xi_1 = 0$ can be determined from the solutions of equations (51) to (56), since

$$B_0 = -\left(\frac{\sigma h_0''(0)}{\lambda\sqrt{2}}\right),$$

$$B_1 = \frac{\sigma}{\lambda\sqrt{2}}\left(\frac{a_2}{a_1}(h_1''(0) - h_0''(0)) + a_1\{h_{10}''(0) - \tfrac{1}{2}h_0''(0)\}\right).$$

For the case of the circular cylinder, where $S(\xi_1) = \cos\xi_1$, and $a_1 = 1$, $a_2 = \frac{1}{6}$, we find that, for $\sigma = 1$,

$$B(\xi_1) = -0{\cdot}1849 - 0{\cdot}0360\xi_1^2 + O(\xi_1^4).$$

I would like to thank Mr E. J. Watson for suggesting this problem, and for his help with the preparation of this paper. I would also like to thank the S.R.C. for the Research Studentship which enabled me to undertake this research.

Appendix I. The solutions of equations (51) to (56)

Solutions of equations (51) *and* (52) *for various values of* σ

The values of $h_0''(0)$, $g_0(0)$ and $g_0'(0)$ for various values of σ are listed below:

σ	$h_0''(0)$	$g_0(0)$	$g_0'(0)$
$\frac{1}{3}$	0·5810	0·2813	−0·1440
0·5	0·4449	0·2778	−0·1340
0·72	0·34175	0·26776	−0·12102
1·0	0·26145	0·25000	−0·10520
1·25	0·21483	0·23457	−0·09345
2·0	0·13593	0·19444	−0·06780
5·0	0·04750	0·11111	−0·02805

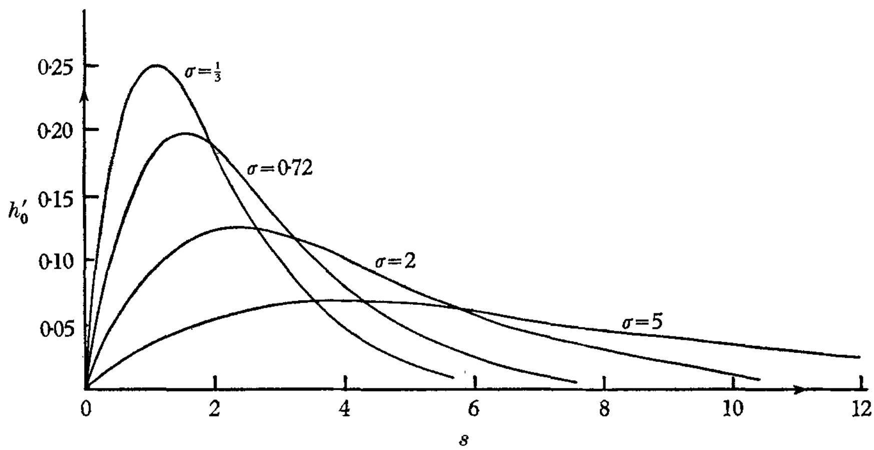

FIGURE 1

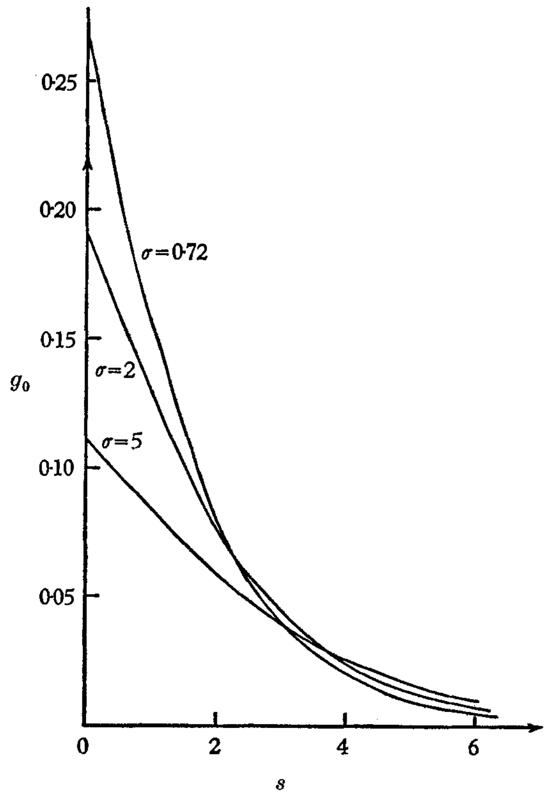

FIGURE 2

Solutions of equations (51) *to* (56) *for the case when* $\sigma = 1$

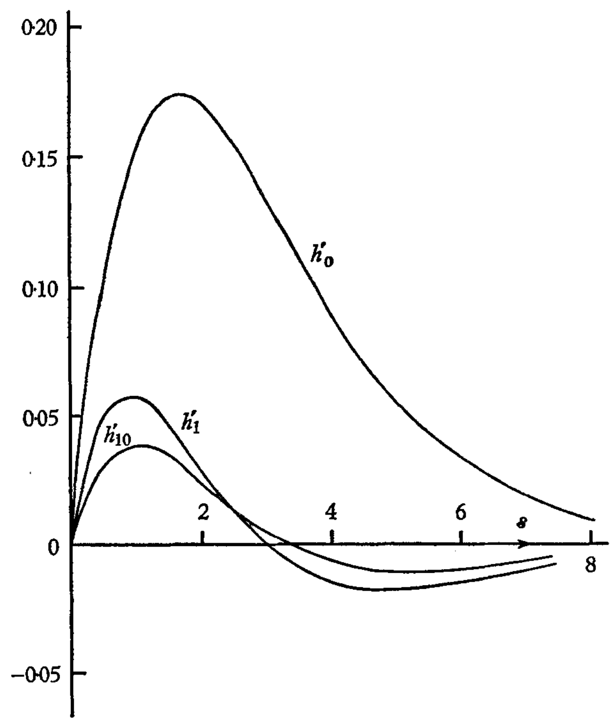

FIGURE 3

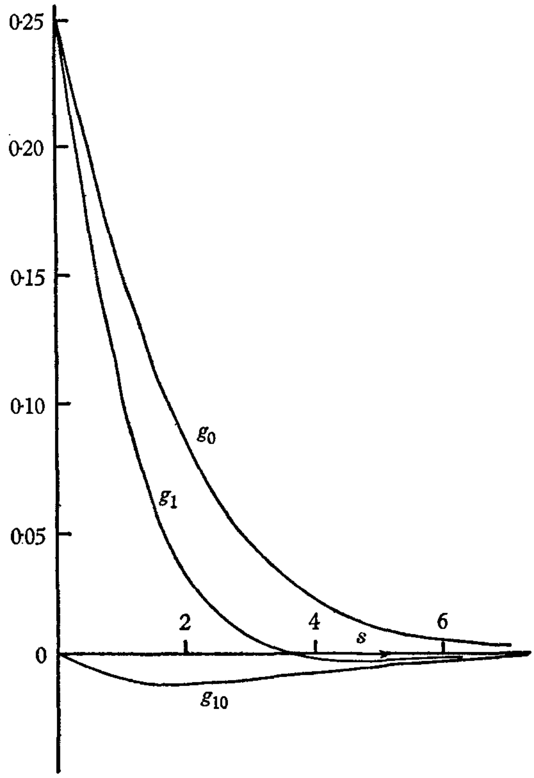

FIGURE 4

Appendix II. To show $A(\xi) \equiv 0$

In the outer boundary layer the nonlinear convective terms are of the same order as the diffusive and buoyancy terms; this suggests defining non-dimensional variables as

$$\frac{T-T_0}{aT_0} = \epsilon^{\frac{1}{5}}\bar{\theta}(\xi,\bar{\eta},\tau),$$

$$\psi + \left(\frac{K}{w}\right)^{\frac{1}{2}}\left(\frac{g\beta aT_0}{w}\right)\frac{S(\xi)}{(1+\sigma^{\frac{1}{2}})}\cos(\tau+\tfrac{1}{4}\pi) = \epsilon^{-\frac{7}{10}}\left(\frac{g\beta aT_0}{w}\right)\left(\frac{K}{w}\right)^{\frac{1}{2}}\bar{f}(\xi,\bar{\eta},\tau),$$

$$\bar{\eta} = \epsilon^{\frac{3}{10}}\left(\frac{w}{K}\right)^{\frac{1}{2}} y,$$

where ξ and τ are given by (4). On putting these in equations (2) and (3), we get equations similar to (34) and (35) with boundary conditions

$$\frac{\partial\bar{f}}{\partial\bar{\eta}} \to 0, \quad \bar{\theta} \to 0 \quad \text{as} \quad \bar{\eta} \to \infty,$$

and the matching condition becomes

$$\bar{\theta} \sim A(\xi)\frac{dS}{d\xi}\frac{\bar{\eta}}{\sqrt{2}} + \dots,$$

$$\bar{f} \sim -S\frac{dS}{d\xi}\frac{A(\xi)\,\bar{\eta}^4}{24\sqrt{2}} + \dots.$$

We now expand $\bar{f}$ and $\bar{\theta}$ in the form

$$\bar{f} = \bar{f}_0 + \epsilon^{\frac{1}{2}}\bar{f}_1 + \epsilon^{\frac{3}{5}}\bar{f}_2 + \dots,$$
$$\bar{\theta} = \bar{\theta}_0 + \epsilon^{\frac{1}{2}}\bar{\theta}_1 + \epsilon^{\frac{3}{5}}\bar{\theta}_2 + \dots.$$

By the same argument as that used in §4, we find that the equations for $\bar{f}_0$ and $\bar{\theta}_0$ are the same as equations (43) and (44), but the matching condition is

$$\bar{\theta}_0 \sim A(\xi)\frac{dS}{d\xi}\frac{\bar{\eta}}{\sqrt{2}},$$
$$\bar{f}_0 \sim -S\frac{dS}{d\xi}\frac{A(\xi)\bar{\eta}^4}{24\sqrt{2}} \quad \text{near} \quad \bar{\eta} = 0.$$

We expand $\bar{f}_0$ and $\bar{\theta}_0$ in series in ξ_1 as in (48) and (49). This yields the system of equations (51) to (56) for the first three terms in each series; the equations for the first term in each series are

$$\sigma h_0''' + g_0 + h_0 h_0'' - h_0'^2 = 0, \tag{57}$$

$$g_0'' + g_0' h_0 - h_0' g_0 = 0. \tag{58}$$

We expand $A(\xi_1)$ in the form

$$A(\xi_1) = A_0 + A_1\xi_1^2 + O(\xi_1^4),$$

so that near $s = 0$

$$h_0 \sim A_0 s^4/24a_1\sqrt{2}, \quad g_0 \sim -A_0 s/\nu\sqrt{2}$$

and $h_0' \to 0$, $g_0 \to 0$ as $s \to \infty$ (where $s = \lambda\bar{\eta}$ and $\lambda^4 = a_1$, as before). From equations (57) and (58), we see that near $s = 0$, h_0 and g_0 can be determined completely by the unknown constant A_0, and asymptotically (for large s) by three unknown constants. Hence equations (57) and (58), which form a fifth-order system, are determined by four unknowns, which cannot be, unless $h_0(s) \equiv 0$, and $g_0(s) \equiv 0$, which in turn means that $A_0 \equiv 0$. Once we have shown that $h_0(s) \equiv 0$ and $g_0(s) \equiv 0$ it follows at once that the other terms in the expansions of $\bar{f}_0$ and $\bar{\theta}_0$ are all identically zero, which means that the $A_i \equiv 0$. From this we can conclude that $A(\xi_1) \equiv 0$.

Appendix III. To show $C \equiv 0$

As before, we want to make the nonlinear convective terms the same order as the diffusive and buoyancy terms, so we introduce non-dimensional variables in the form

$$\frac{T - T_0}{aT_0} = \epsilon^{\frac{4}{5}}\bar{\theta}(\xi, \bar{\eta}, \tau),$$

$$\psi + \left(\frac{K}{w}\right)^{\frac{1}{2}}\left(\frac{g\beta aT_0}{w}\right)\frac{S(\xi)}{(1+\sigma^{\frac{1}{2}})}\cos\left(\tau + \tfrac{1}{4}\pi\right) = \epsilon^{-\frac{3}{5}}\left(\frac{g\beta aT_0}{w}\right)\left(\frac{K}{w}\right)^{\frac{1}{2}}\bar{f}(\xi, \bar{\eta}, \tau),$$

$$\bar{\eta} = \epsilon^{\frac{2}{5}}\left(\frac{w}{K}\right)^{\frac{1}{2}} y,$$

and ξ and τ are given by (4). On substituting in (2), (3) we get equations similar to (34) and (35). An expansion of $\bar{f}$ and $\bar{\theta}$ in series in ϵ gives equations (43) and (44) as those satisfied by the first terms $\bar{f}_0$ and $\bar{\theta}_0$ in each series. The matching condition is now

$$\bar{\theta}_0 \sim \frac{dS}{d\xi}\frac{C\bar{\eta}}{\sqrt{2}},$$

$$\bar{f}_0 \sim -S\frac{dS}{d\xi}\frac{C\bar{\eta}^4}{24\sqrt{2}}.$$

This then gives a situation analogous to that in appendix II, and we can then conclude that $C = 0$.

REFERENCES

GOLDSTEIN, S. 1938 (editor) *Modern Developments in Fluid Mechanics*, Vol. II. Oxford.

LIGHTHILL, M. J. 1963 Real and ideal fluids. Chapter I in *Laminar Boundary Layers* (ed. L. Rosenhead), pp. 1–45. Oxford.

RILEY, N. 1965 Oscillating viscous flows. *Mathematika*, **12**, 161–75.

SCHLICHTING, H. 1932 Berechnung ebner periodischer Grenzschichtströmungen. *Phys. Z.* **33**, 327–35.

STUART, J. T. 1963 Unsteady boundary layers. Chapter VII in *Laminar Boundary Layers* (ed. L. Rosenhead), pp. 349–408. Oxford.

STUART, J. T. 1966 Double boundary layers in oscillatory viscous flow. *J. Fluid Mech.* **24**, 673–87.

WHITHAM, G. B. 1963 The Navier–Stokes equations of motion. Chapter III in *Laminar Boundary Layers* (ed. L. Rosenhead). Oxford.